

Double positive solutions of boundary value problems for p -Laplacian impulsive functional dynamic equations on time scales

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Abstract

This paper is devoted to studying the existence of positive solutions of the boundary value problem for p -Laplacian impulsive functional dynamic equations on time scales. Existence results of at least two positive solutions are established via a fixed point theorem in a cone.

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1. Introduction

Let T be a closed nonempty subset of R , and let T have the subspace topology inherited from the Euclidean topology on R . In some of the current literature, T is called a time scale (or measure chain). For notation, we shall use the convention that, for each interval J of R , J will denote the time scale interval, that is, $J := J \cap T$.

There is currently much activity focused on time scales. Included in this activity, the theory of dynamic equations on time scales has received much attention in recent years. This theory unifies existing results in differential and finite difference equations, and provides powerful new tools for exploring connections between the traditionally separated fields. We refer to the books by Bohner and Peterson [9], Lakshmikantham et al. [12], the papers [1,3,8,10] and the papers cited therein.

On the other hand, impulsive differential equations, which arise in physics, population dynamics, economics, and so on (see [5] and references therein), have become more and more important in mathematical models of real processes. And the boundary value problems (hereafter to be abbreviated as BVPs) for impulsive differential equations and impulsive difference equations (see [2]) have received special attention from many authors in recent years. However, there is not much reported concerning the boundary value problems for impulsive dynamic equations on time scales [5–7,11]. In 2002, Henderson [11] discussed the boundary value problem for second-order impulsive

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dynamic equations on a time scale T

$$\begin{cases} y^{\Delta\Delta}(t) + f(y(\sigma(t))) = 0, & t \in [0, 1]_T \setminus \{\tau\} \\ \text{Imp}(y(\tau)) = I(y(\tau)) \\ y(0) = y^{\Delta}(\sigma(1)) = 0 \end{cases} \quad (1.1)$$

where $\text{Imp}(y(\tau)) = y(\tau^+) - y(\tau^-)$, $f \in C(R, R^+)$, σ is the forward jump operator, and $I \in C(R^+, R^+)$. The author established the existence of at least two positive solutions of (1.1) via a double fixed point theorem.

Motivated by the above results, in this paper, we are concerned with the existence of positive solutions of the boundary value problem for the following p -Laplacian impulsive functional dynamical equations on a time scale

$$\begin{cases} [\phi_p(y^{\Delta}(t))]^{\nabla} + a(t)f(y(t), y(\mu(t))) = 0 & t \in [0, 1]_T \setminus \{\tau\} \\ \text{Imp}(y(\tau)) = I(y(\tau)) \\ y_0(t) = \varphi(t), & t \in [-r, 0] \\ y(0) = y^{\Delta}(1) \end{cases} \quad (1.2)$$

where $-r, 0, 1 \in T$; T is a time scale. $\phi_p(s)$ is p -Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$; $\tau \in (0, 1)_T$ and

(A₁) $\text{Imp}(y(\tau)) = y(\tau^+) - y(\tau^-)$, $I \in C(R^+, R^+)$;

(A₂) $f : (R^+)^2 \rightarrow R^+$ is continuous (R^+ denotes the nonnegative reals);

(A₃) $a : T \rightarrow R^+$ is left dense continuous (i.e., $a \in C_{\text{ld}}(T, [0, +\infty))$) and does not vanish identically on any closed subinterval of $[0, 1]_T$, where $C_{\text{ld}}(T, R^+)$ denotes the set of all left dense continuous functions from T to R^+ ;

(A₄) $\varphi : [-r, 0] \rightarrow R^+$ is continuous and $r > 0$;

(A₅) $\mu : [0, 1] \rightarrow [-r, 1]$ is continuous and $\mu(t) \leq t$ for all t .

We remark that by a solution y of (1.2), we mean $y : T \rightarrow R$ is delta differentiable, $y^{\Delta} : T^k \rightarrow R$ is nabla differentiable on $T^k \cap T_k$ and $y^{\Delta\nabla} : T^k \cap T_k \rightarrow R$ is continuous, and y satisfies the impulsive and boundary conditions of (1.2). If $y^{\Delta\nabla}(t) \leq 0$ on $[0, 1]_{T^k \cap T_k}$, then we say y is concave on $[0, 1]_T$.

By using a fixed point theorem due to Avery and Henderson [4], we prove that there exist at least two positive solutions of problem (1.2). To this end, in Section 2, we provide some background materials from the theory of cones in Banach spaces, and then we state the double fixed point theorem. In Section 3, by defining an appropriate Banach space and a cone, we impose the growth conditions on f , I and φ which allow us to apply the double fixed point theorem in obtaining existence of two positive solutions of (1.2). To the authors' best knowledge, the question of positive solutions of boundary value problems for impulsive functional dynamics equations on time scales has not been studied. The main results in this paper generalize the paper [11].

For convenience, we list the following well-known definitions which can be found in [9] and the references therein.

Definition 1.1. For $t < \sup T$ and $t > \inf T$, we define the forward jump operator, σ , and the backward jump operator, ρ , respectively, by

$$\sigma(t) = \inf\{\tau \in T | \tau > t\} \in T, \quad \rho(t) = \sup\{\tau \in T | \tau < t\} \in T \quad \text{for all } t \in T.$$

If $\sigma(t) > t$, t is said to be right scattered, and if $\sigma(t) = t$, t is said to be right dense (rd). If $\rho(t) < t$, t is said to be left scattered, and if $\rho(t) = t$, t is said to be left dense (ld). If T has a right scattered minimum m , define $T_k = T - \{m\}$; otherwise set $T_k = T$. If T has a left scattered maximum M , define $T^k = T - \{M\}$; otherwise set $T^k = T$.

Definition 1.2. For $y : T \rightarrow R$ and $t \in T^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|y(\sigma(t)) - y(s) - y^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. For $y : T \rightarrow R$ and $t \in T_k$, we define the nabla derivative of $y(t)$, $y^{\nabla}(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U' of t such that

$$|y(\rho(t)) - y(s) - y^{\nabla}(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U'$.

If f is delta differential at t , then f is continuous at t . If $T = \mathbb{R}$ then $f^\Delta(t) = f'(t)$. If $T = \mathbb{Z}$ then $f^\Delta(t) = f(t+1) - f(t)$ is the forward difference operator while $f^\nabla(t) = f(t) - f(t-1)$ is the backward difference operator.

Definition 1.3. If $F^\Delta(t) = f(t)$ for each $t \in T^k$, then we define the delta integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a),$$

and F is called antiderivative of f . If $F^\nabla(t) = f(t)$ for each $t \in T_k$, then we define the delta integral by

$$\int_a^t f(s) \nabla s = F(t) - F(a)$$

and F is called antiderivative of f .

2. Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces, and we then state the triple fixed point theorem for a cone preserving operator.

Definition 2.1. Let $E = (E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a cone, if it satisfies the following two conditions:

- (i) $u \in P, \lambda \geq 0$ implies $\lambda u \in P$; and
- (ii) $u \in P, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2.2. Given a cone P in a real Banach space E , a functional $\psi : P \rightarrow \mathbb{R}$ is said to be increasing on P , provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

For a nonnegative continuous functional γ on a cone P in a real Banach space E , and each $d > 0$, we set

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\}.$$

The key tool to obtain our main result is the following double fixed point theorem due to Avery et al. [4].

Theorem 2.1. Let P be a cone in a real Banach space E and α and γ be increasing, nonnegative, continuous functionals on P , and let θ be a nonnegative, continuous functional on P with $\theta(0) = 0$ such that for some $c > 0$ and $M > 0$,

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq M\gamma(x),$$

for all $x \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ such that

$$\theta(\lambda x) \leq \lambda \theta(x) \quad \text{for } 0 \leq \lambda \leq 1 \text{ and } x \in \partial P(\theta, b)$$

and

$$F : \overline{P(\gamma, c)} \rightarrow P$$

is a completely continuous operator such that:

- (B₁) $\gamma(Fx) > c$, for all $x \in \partial P(\gamma, c)$;
- (B₂) $\theta(Fx) < b$, for all $x \in \partial P(\theta, b)$;
- (B₃) $P(\alpha, a) \neq \emptyset$, and $\alpha(Fx) > a$, for all $x \in \partial P(\alpha, a)$.

Then F has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ such that

$$a \leq \alpha(x_1) \quad \text{with } \theta(x_1) < b.$$

And

$$b < \theta(x_2) \quad \text{with } \gamma(x_2) < c.$$

3. Positive solutions

In this section, we impose growth conditions on f , I and φ , and then apply [Theorem 2.1](#) to establish the existence of two positive solutions of boundary value problem (1.2). We note that, from the nonnegativity of f and (A_3) , a solution of BVP (1.2) is nonnegative and concave on $[0, 1]$. To apply [Theorem 2.1](#), we must define a suitable Banach space, E , a cone P , and an operator F . In that direction, let

$$E = \{y : [0, 1]_T \rightarrow R \mid y \in C[0, \tau]_T, y \in C(\tau, 1]_T \text{ and } y(\tau^+) \in R\}$$

equipped with norm $\|y\| = \max\{\sup_{t \in [0, \tau]_T} |y(t)|, \sup_{t \in (\tau, 1]_T} |y(t)|\}$, and define a cone $P \subset E$ by

$$P = \{y \in E \mid y \text{ is concave, nondecreasing and nonnegative on each of } [0, \tau]_T \text{ and } [\tau, 1]_T \text{ and } \text{Imp}(y(\tau)) \geq 0\}.$$

For each $y \in P$, $I(y(\tau)) \geq 0$. It follows that, for $y \in P$,

$$\|y\| = \max\{y(\tau), y(1)\} = y(1).$$

For the remainder, assume there exists

$$\eta = \inf \left[\frac{\tau + 1}{2}, 1 \right)_T \in T,$$

and fix $0 < \eta < l < 1$.

In addition, if $y \in P$, then

$$y(t) \geq \frac{1}{2} \sup_{s \in [\frac{\tau}{2}, \tau]} y(s) = \frac{1}{2} y(\tau), \quad t \in \left[\frac{\tau}{2}, \tau \right]_T,$$

and

$$y(t) \geq \frac{1}{2} \sup_{s \in [\eta, 1]_T} y(s) = \frac{1}{2} y(1), \quad t \in [\eta, 1]_T.$$

We define the increasing, nonnegative, continuous functionals γ , θ , and α on P , by

$$\gamma(y) = \min_{t \in [\eta, l]_T} y(t) = y(\eta), \quad \theta(y) = \min_{t \in [\eta, l]_T} y(t) = y(\eta)$$

and

$$\alpha(y) = \max_{t \in [\tau, l]_T} y(t) = y(l).$$

Then, for each $y \in P$,

$$\gamma(y) = \theta(y) \leq \alpha(y),$$

and

$$\gamma(y) = y(\eta) \geq \frac{1}{2} y(1) = \frac{1}{2} \|y\|.$$

Hence,

$$\|y\| \leq 2\gamma(y), \quad \text{for all } y \in P.$$

Moreover, we have that

$$\theta(\lambda y) \leq \lambda \theta(y) \quad \text{for } 0 \leq \lambda \leq 1 \text{ and } y \in \partial P(\theta, b).$$

For notational convenience, we set

$$K_1 = \eta \phi_q \left(\int_{L_3} a(r) \nabla r \right), \quad K_2 = \eta \phi_q \left(\int_0^1 a(r) \nabla r \right), \quad K_3 = l \phi_q \left(\int_{L_3} a(r) \nabla r \right),$$

$$L_1 = \{t \mid \mu(t) < 0, t \in [0, 1]_T\}, \quad L_2 = \{t \mid \mu(t) \geq 0, t \in [0, 1]_T\}, \quad L_3 = L_1 \cap [l, 1]_T$$

and assume $L_3 \neq \phi$, $\int_{L_3} a(r) \nabla r > 0$.

We note that $y(t)$ is a solution of (1.2) if and only if

$$y(t) = \begin{cases} I(y(\tau))\chi_{(\tau,1]_T}(t) + \int_0^t \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s, & t \in [0, 1]_T \\ \varphi(t), & t \in [-r, 0] \end{cases} \quad (3.1)$$

where $\chi_{(\tau,1]_T}(t)$ is the characteristic function. For each $y \in P$, extend $y(t)$ to $[-r, 1]$ with $y(t) = \varphi(t)$ for $t \in [-r, 0]$.

Theorem 3.1. Suppose that there exist positive numbers $a < b < c$ such that

$$0 < a < \frac{K_3}{K_2}b < \frac{K_3}{2K_2}c,$$

and assume that f , I and φ satisfy the following conditions:

(C₁) $f(w, \varphi(s)) > \phi_p(\frac{c}{K_1})$ for $c \leq w \leq 2c$, uniformly in $s \in [-r, 0]$;

(C₂) $f(w, \varphi(s)) < \phi_p(\frac{b}{K_2})$ for $0 \leq w \leq 2b$, uniformly in $s \in [-r, 0]$ and

$$f(w_1, w_2) < \phi_p \left(\frac{b}{K_2} \right) \quad \text{for } 0 \leq w_i \leq 2b, \quad i = 1, 2;$$

(C₃) $f(w, \varphi(s)) > \phi_p(\frac{a}{K_3})$ for $a \leq w \leq 2a$, uniformly in $s \in [-r, 0]$;

(C₄) $I(w) \leq \frac{b}{2}$, if $0 \leq w \leq b$.

Then, BVP (1.2) has at least two positive solutions of the form

$$y(t) = \begin{cases} y_i(t), & t \in [0, 1]_T, i = 1, 2, \\ \varphi(t), & t \in [-r, 0] \end{cases}$$

where $a < \alpha(y_1)$, $\theta(y_1) < b$ and $b < \theta(y_2)$, $\gamma(y_2) < c$.

Proof. Define $F : P \rightarrow E$ as

$$Fy(t) = I(y(\tau))\chi_{(\tau,1]_T}(t) + \int_0^t \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s, \quad t \in [0, 1]_T.$$

We need to seek fixed points of F in the cone P . To this end, it suffices to show that the conditions of Theorem 2.1 hold with respect to F .

Firstly, let $y \in \overline{P(\gamma, c)}$. By the nonnegativity of $a(t)$, f and I , for $t \in [0, 1]_T$, $Fy(t) \geq 0$. Moreover, $(Fy)^{\Delta \nabla}(t) = -a(t)f(y(t), y(\mu(t))) \leq 0$, for $t \in [0, 1]_{T^k \cap T_k} \setminus \{\tau\}$, which implies $Fy(t)$ is concave on each of $[0, \tau]_T$ and $[\tau, 1]_T$. In addition,

$$(Fy)^{\Delta}(t) = \phi_q \left(\int_t^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \geq 0$$

so that $Fy(t)$ is nondecreasing on each of $[0, \tau]_T$ and $[\tau, 1]_T$. Since $Fy(0) = 0$, we have $Fy(t) \geq 0$ on $[0, \tau]_T$. Also, since $y \in \overline{P(\gamma, c)}$,

$$\text{Imp}(Fy(\tau)) = Fy(\tau^+) - Fy(\tau) = I(y(\tau)) \geq 0.$$

This yields $Fy(\tau^+) \geq Fy(\tau) \geq 0$, and, consequently, $Fy(t) \geq 0$ on $[\tau, 1]_T$. Therefore, we have $F : \overline{P(\gamma, c)} \rightarrow P$.

Next, we shall demonstrate that F is continuous and completely continuous. The proof is divided into three steps.
Step 1. To show the continuity of F .

Let $\{y_n\}_{n=1}^\infty$ be a sequence such that $\lim_{n \rightarrow \infty} y_n = y$ in P . Then

$$|Fy_n(t) - Fy(t)| \leq \sup_{t \in [0,1]_T \setminus \{\tau\}} \left| \int_0^t \phi_q \left(\int_s^1 a(r) f(y_n(r), y_n(\mu(r))) \nabla r \right) - \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \right| \Delta s + |I(y_n(\tau)) - I(y(\tau))|.$$

As f , ϕ_q and I are continuous, it follows that $\|Fy_n - Fy\| \rightarrow 0$ as $n \rightarrow \infty$. That is, F is continuous.

Step 2. To show that F maps bounded sets into bounded sets in P .

By the continuity of f , ϕ_q and I , one can easily derive that there is a constant $C > 0$ such that

$$|Fy(t)| \leq |I(y(\tau))| + \sup_{t \in [0,1]_T} \left| \int_0^t \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \right| \leq C$$

is always valid for any $y \in U_\delta = \{y \in P \mid \|y\| \leq \delta\}$.

Step 3. To show that F is equicontinuous in P .

Let $t_1, t_2 \in [0, 1]_T$, $y \in U_\delta$, then

$$|Fy(t_1) - Fy(t_2)| \leq \phi_q \left(\int_0^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) |t_1 - t_2|.$$

Clearly, the right-hand side tends uniformly to zero when $|t_1 - t_2| \rightarrow 0$.

By the above arguments and the Arzela–Ascoli Theorem, we have that $F : P \rightarrow P$ is continuous and completely continuous.

We now verify that the condition (B₁) of Theorem 2.1 is satisfied.

Taking $y \in \partial P(\gamma, c)$, we have $\gamma(y) = y(\eta) = c$. Then $y(t) \geq c$, for $t \in [\eta, 1]_T$. Recalling that $\|y\| \leq 2\gamma(y) = 2c$, we get

$$c \leq y(t) \leq 2c, \quad t \in [\eta, 1]_T.$$

So

$$\begin{aligned} \gamma(Fy) &= Fy(\eta) = I(y(\tau))\chi_{(\tau,1]_T}(\eta) + \int_0^\eta \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \\ &= I(y(\tau)) + \int_0^\eta \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \\ &\geq \eta \phi_q \left(\int_l^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \\ &\geq \eta \phi_q \left(\int_{L_3} a(r) f(y(r), \varphi(\mu(r))) \nabla r \right) \\ &\geq \eta \phi_q \left(\int_{L_3} a(r) \nabla r \right) \frac{c}{K_1} = c. \end{aligned}$$

Then, we turn to the condition (B₂) of Theorem 2.1. We choose $y \in \partial P(\theta, b)$, then $\theta(y) = y(\eta) = b$, and $0 \leq y(t) \leq b$, for $t \in (\tau, \eta]_T$. Since $y \in P$ implies $y(\tau) \leq y(\tau^+)$, and also $y(t)$ is nondecreasing on $[0, \tau]_T$, we have

$$y(t) \leq b, \quad t \in [0, \tau]_T,$$

and so by the assumption (C₄), it follows that

$$I(y(\tau)) \leq \frac{b}{2}.$$

Note that $\|y\| \leq 2\gamma(y) \leq 2\theta(y) = 2b$, then we have

$$0 \leq y(t) \leq 2b, \quad t \in [0, 1]_T.$$

Hence

$$\begin{aligned} \theta(Fy) &= Fy(\eta) = I(y(\tau))\chi_{(\tau, 1]_T}(\eta) + \int_0^\eta \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \\ &= I(y(\tau)) + \int_0^\eta \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \\ &\leq \frac{b}{2} + \eta \phi_q \left(\int_0^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \\ &= \frac{b}{2} + \eta \phi_q \left(\int_{L_1} a(r) f(y(r), \varphi(\mu(r))) \nabla r + \int_{L_2} a(r) f(y(r), y(\mu(r))) \nabla r \right) \\ &\leq \frac{b}{2} + \frac{\eta b}{2K_2} \phi_q \left(\int_0^1 a(r) \nabla r \right) = b. \end{aligned}$$

Finally, we show that the condition (B₃) of Theorem 2.1 is satisfied. It is obvious that $P(\alpha, a) \neq \phi$. On the other hand, for $y \in \partial P(\alpha, a)$, we have $\alpha(y) = y(l) = a$. Hence,

$$a \leq \alpha(y) \leq 2a, \quad t \in [l, 1]_T.$$

Thus,

$$\begin{aligned} \alpha(Fy) &= Fy(l) = I(y(\tau))\chi_{(\tau, 1]_T}(l) + \int_0^l \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \\ &= I(y(\tau)) + \int_0^l \phi_q \left(\int_s^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \Delta s \\ &\geq l \phi_q \left(\int_l^1 a(r) f(y(r), y(\mu(r))) \nabla r \right) \\ &\geq l \phi_q \left(\int_{L_3} a(r) f(y(r), \varphi(\mu(r))) \nabla r \right) \\ &\geq l \phi_q \left(\int_{L_3} a(r) \nabla r \right) \frac{a}{K_3} = a. \end{aligned}$$

By Theorem 2.1, F has at least two fixed points y_1 and y_2 satisfying

$$a < \alpha(y_1), \quad \theta(y_1) < b \quad \text{and} \quad b < \theta(y_2), \quad \gamma(y_2) < c.$$

Now, we set

$$y(t) = \begin{cases} y_i(t), & t \in [0, 1]_T, \quad i = 1, 2, \\ \varphi(t), & t \in [-r, 0] \end{cases}$$

which are two positive solutions of the boundary value problem (1.2). The proof is complete. \square

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